Complex Analysis I (MATH 6460)

Lab 1
Exponential, hyperbolic and trigonometric functions

Type "http://www.math.tamu.edu/˜romwell/CGApplet/" in your browser (or just click on "Link to Complex Grapher" on the class web-page). We will use a java applet for Complex Grapher, a courtesy of Prof. Andrew Bennett, Kansas State University.

Note: In this lab you may need to look at larger plots. You can change the size of the plot in all 3 dimension by right-clicking on the graph.

1. Exponential function. Graph a function $f(z) = \exp(z)$. This function's properties were already discussed, so you will need just confirm them by graphing. This part is not for credit.

(a) Does $\exp(z)$ have any zeroes? How do know it from the picture?

(b) Is this function periodic? What is its period? How does the graph support your statements?

(c) How does $\exp(z)$ behave along real axis? Along imaginary axis? Along lines parallel to either real or imaginary axes?

Now we will discuss the properties of the complex sine, cosine, hyperbolic sine and cosine functions. Their definitions are:

$$\sinh(z) := \frac{e^z - e^{-z}}{2} \qquad \sin(z) := \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cosh(z) := \frac{e^z + e^{-z}}{2} \qquad \cos(z) := \frac{e^{iz} + e^{-iz}}{2}$$

For all the questions in the rest of the lab, make a conjecture based on the picture you see, and then **prove** it algebraically, using the definition.

2. Hyperbolic functions. Graph the complex hyperbolic sine function $f(z) = \sinh(z)$. Recall that **real** sinh function is always increasing, not periodic, odd, and has unique zero at 0.

(a) How many zeroes does $\sinh(z)$ appear to have? Find all zeroes of $\sinh(z)$.

(b) Does $\sinh(z)$ appear to be periodic? What is its period?

(c) Is complex sinh function even, odd or neither?

(d) Graph the complex hyperbolic cosine function $f(z) = \cosh(z)$. Recall that **real** cosh function is not periodic, even, always positive and thus has no zeros. Now answer the above questions (a)-(c) for $\cosh(z)$.

3. Trigonometric functions. Graph the complex sine and cosine functions: $f(z) = \sin(z)$ and $f(z) = \cos(z)$. Recall that **real** sine and cosine functions are bounded, periodic with period 2π, have infinitely many zeroes; and $\sin(x)$ is odd, while $\cos(x)$ is even.

(a) How many zeroes in the complex plane does $\sin(z)$ appear too have? Find all zeroes of $\sin(z)$.

(b) Does $\sin(z)$ appear to be periodic? What is its period?

(c) Is $\sin(z)$ bounded? (Note that complex-valued function $f(z)$ is called bounded if $|f(z)|$ is bounded.)

(d) Is complex sine function even, odd or neither?

(e) Now answer the above questions (a)-(d) for $\cos(z)$.

4. Identities. Graph the functions below. Based on the graph, guess which familiar function the graph resembles. Prove your conjectures.

(a) $f(z) = \cosh(z) + \sinh(z)$

(b) $f(z) = \cosh^2(z) - \sinh^2(z)$

(c) $f(z) = \cosh(iz)$

(d) $f(z) = \cos(iz)$

(e) $f(z) = \sinh(iz)$

(f) $f(z) = \sin(iz)$

5. Series. One of the ways to explain the close relationship between exponential and trigonometric functions is through Taylor series expansion. Using known Taylor series from the real case, "prove" that

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

(We will see later on, in Chapter V, that these series expansion indeed make sense in the complex case.)