Dynamical Systems (MATH 3410)

Lab 2 - Rates of Convergence

This lab is based on Experiment 5.6 from the textbook (p.48).

You may use Mathematica notebook files I have uploaded on the class web-page at `http://math-cs.cns.uni.edu/~ostapyuk/MathematicaLabs.html` or your own programs.

Consider the following functions:

(a) $F(x) = x^2 + 0.25$

(b) $F(x) = x^2$

(c) $F(x) = x^2 - 0.24$

(d) $F(x) = x^2 - 0.75$

(e) $F(x) = x(1 - x)$

(f) $F(x) = 0.4x(1 - x)$

(g) $F(x) = 1.6x(1 - x)$

(h) $F(x) = 2x(1 - x)$

(i) $F(x) = 2.4x(1 - x)$

(j) $F(x) = 3x(1 - x)$

(k) $F(x) = 0.4 \sin x$

(l) $F(x) = \sin x$.

Each of those functions has an attracting or neutral fixed point such that an orbit of 0.2 converges to it.

1. For each of these functions, find an attracting or neutral fixed point. You have to find exact values solving equations (for the last two functions, make sure there are no other fixed points by graphing them). For each of the functions, record the fixed point, the derivative at the fixed point, and the number of iterations it takes for the orbit of 0.2 to reach the fixed point within $\varepsilon = 0.001$. Present your findings in the form of a table.

Note: if Overflow error occurs or the program keeps running for a long time (more than a few minutes), one of the following happens:

1. The value pf is not an attracting or neutral fixed point; check your computations.

2. The fixed point is too far away from the initial point 0.2, so the orbit of 0.2 does not converge to it. (This won't happen for the functions given; but may happen if you consider your own examples.)

3. The orbit of 0.2 does converge to the fixed point, but it takes too many iterations to confirm this. Change the accuracy level.

2. What is the relationship between the speed of convergence and the derivative at the fixed point? Which fixed points have fastest convergence and which have slowest? What can be said about neutral fixed points? (Note that since the orbits converge to the neutral points, all neutral points we found are either weakly attracting or one-side attracting.)

3. Now consider the following functions that have either neutral or attracting 2-cycles:

(A) $F(x) = x^2 - 1$

(B) $F(x) = x^2 - 1.1$

(C) $F(x) = x^2 - 1.25$

Compute the cycle exactly (for parts (A) and (C)) or numerically (for part (B); use graphing calculator or Mathematica program NumSolutions.nb). For numerical values, enter as many digits as you obtain. Find the derivative of the second iterate of F at the points of the cycle. Compute the orbit of 0.2 until it gets close enough (within $\varepsilon = 0.001$) to the cycle. Record number of iterates. Present your findings in the form of a table, similarly to the question 1.

4. Again, discuss how much the derivative of $F^{\circ 2}$ affects the rate of convergence and what is the difference between attracting and neutral cycles.

5. Some of the fixed points and cycles you found above have derivative of F (respectively, of $F^{\circ 2}$ for the 2-cycles) equal to zero. Such points and cycles are called "super-attracting". List these points and cycles (together with corresponding functions) here. Look at the rates of convergence you found above. Increase the accuracy level (i.e. make ε smaller) and check the rate of convergence again. Explain why the term "super-attracting" makes sense.

Note: An experiment like this is not a perfect tool to measure the rates of convergence of iterates. To fully discuss the relationship between the rates and the derivatives, you will need to consider the dependence of convergence on ε and the distance from the seed (initial point) to the fixed point. However, the experiment should give some ideas on the subject.