Dynamical Systems (MATH 3410)

Lab 4 - Feigenbaum's Constant

This lab is based on Experiment 10.4 from the textbook (p.128).

Using the orbit diagram, we have discovered that the quadratic family $Q_c(x) = x^2 + c$ undergoes a sequence of period-doubling bifurcations as the parameter c decreases. Moreover, under magnifications the parts of the orbit diagram look very similar. In this lab, we will show that these period-doubling bifurcations indeed occur at the same rate.

We have found that the quadratic family has the unique critical point $x_0 = 0$. You will need to find c-values at which the critical point 0 lies on an attracting cycle of the prime period 2^n for $n = 0, 1, \ldots, 6$.

1. For $n = 0$ and $n = 1$, we have derived the formulas for fixed points and 2-cycles of $Q_c(x) = x^2 + c$ (check section 6.1 if you forgot), so c-values c_0 and c_1 can we found exactly. Find them by assuming that $x_0 = 0$ is an attracting fixed point (lies on an attracting 2-cycle, respectively).

2. For $n \geq 2$, you will have to find c_n approximately, accurate to 7 decimal places.

Suggested procedure:

(a) Use program Full_orbit_diagram.nb to get the first approximation of c-values. Change the value of $IterSkip$ to 0 to show the full orbit behavior, not just asymptotic behavior. Also change the value of $NewIter$ to 200 to get more accurate picture. You will notice that at certain values of c the diagram contains only $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $\ldots$, $2^6 = 64$ points, and one of these point must fall precisely on c-axis. Estimate the values of these c by zooming in, if necessarily. Recall that you can zoom by changing "PlotRange" values in "ListPlot" command. You must be able to approximate c-values up to $2 - 3$ decimal points.

(b) After you get your rough estimate of c_n, use program Orbit.nb to improve the accuracy of the results up to 7 decimal places. Compute the orbit of the critical point 0 under the function $x^2 + c_n$, where c_n is your approximation from the part (a). First, make sure the orbit looks like a 2^n cycle with one of the values very close to 0. If it does not, go back and recheck your values from the part (a). I suggest computing a large enough power of 2 (e.g. 256) of points on the orbit, so the value that is close to 0 will be the last one and easy to notice. Now change the value of c_n in the definition of the function by small increments to get the point on the orbit even closer to zero. Proceed until you get at least 7 decimal places of c_n.

Notes:

- You may use your own algorithm and/or your own program to estimate c-values.
- It may take a long time to compute all c-values. You may work in groups and divide the workload.

- A single mistake in one of the c-values will ruin the final result. I highly recommend to compare your or your group values with somebody else's.

3. Record your data in the form of the table: enter $n = 0, 1, \ldots, 6$ in the first column and write down your exact or approximate values c_0, c_1, ..., c_6 in the second column.

4. Now compute the ratios of the distances between c-values, i.e.

$$f_0 = \frac{c_0 - c_1}{c_1 - c_2}, \quad f_1 = \frac{c_1 - c_2}{c_2 - c_3}, \quad \ldots, \quad f_4 = \frac{c_4 - c_5}{c_5 - c_6}.$$

Make sure to keep at least 7 decimal places. Present the values in the form of a table.

5. Do you notice any convergence? Estimate the value of the limit. This number is called **Feigenbaum's constant**.

6. Now do all of the above for the logistic function family $F_c(x) = cx(1-x)$ and its only critical point $x_0 = 1/2$. Note that since we don't have a formula for a 2-cycle of this family, you will need to estimate c_1 as well.

Compare the value of the limit of f_n's with the result for quadratic family from part **5**.