Dynamical Systems (MATH 3410/5410)

Midterm Exam, Spring 2018

Name: Solutions

Note: unless asked to estimate, compute all values exactly, do not use decimal approximations.

1. (20 pts) Consider the function $f(x) = \frac{4}{x}$.

(a) Find the second and the third iterates of f as functions of x. Find the general form of $f^{\circ n}(x)$.

$f(f(x)) = \frac{4}{4/x} = x$, $f^{\circ 3}(x) = 4/x$

$$f^{\circ n}(x) = \begin{cases} x, & n \text{ is even} \\ 4/x, & n \text{ is odd} \end{cases}$$

(b) Find all fixed points of f and classify them as attracting, repelling, or neutral.

$4/x = x \quad x^2 = 4$, $x = \pm 2$, two fixed points

$f'(x) = -4/x^2 \quad f'(\pm 2) = -4/4 = -1$, both are neutral

(c) Does f have any 2-cycles? Find/describe all of them or show that there are none.

Solve for $x = x$, solutions are all real x.

Two cycles are of the form $\{x, 4/x\}$, $x \in \mathbb{R} \setminus \{0, \pm 2\}$

Infinitely many 2-cycles.

2. **(20 pts)** Consider the function

$$h(x) = \begin{cases} 2x, & x < 1; \\ 2 - x, & 1 \le x. \end{cases}$$

(a) Compute the orbits of $x = 0.25$ and $x = 0.8$ under this function. Describe the fate of each orbit.

$0.25 \to 0.5 \to 1 \to 1 \to 1 \to ..$ eventually fixed

$0.8 \to 1.6 \to 0.4 \to 0.8 \to$ 3-cycle

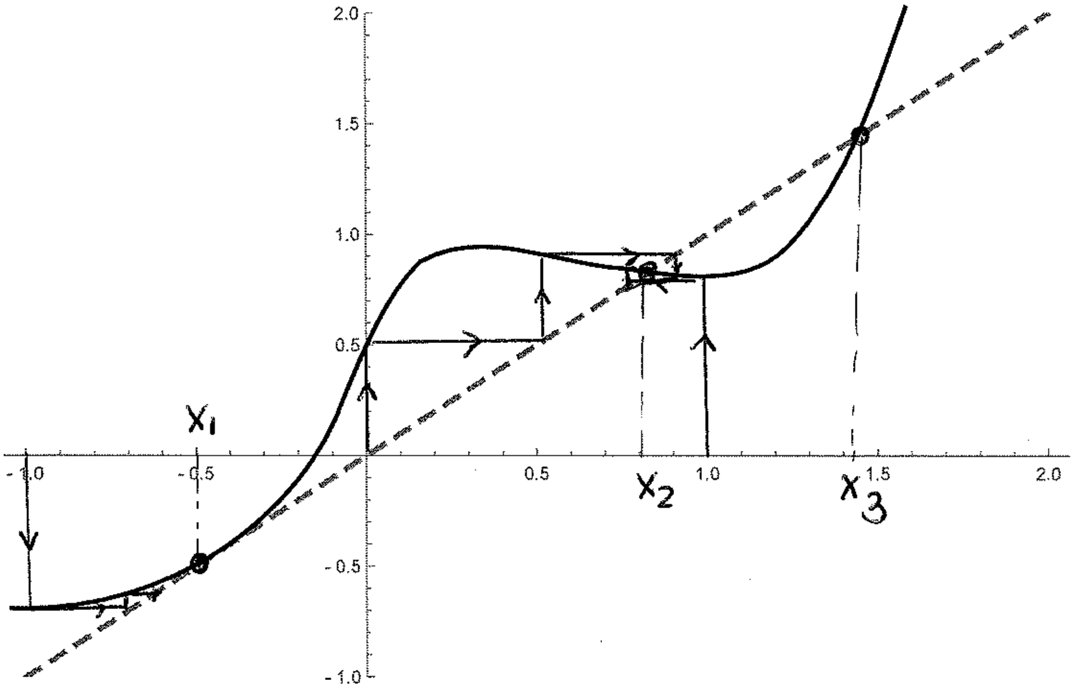
(b) One of the two orbits above is an n-cycle. What is n? Classify this cycle as attracting, repelling, or neutral.

$n = 3$

$$h'(x) = \begin{cases} 2, & x < 1 \\ -1, & x > 1 \end{cases}$$

$h'(0.8) \cdot h'(1.6) \cdot h'(0.4) = 2(-1) \cdot 2 = -4$

repelling cycle



3. **(20 pts)** Consider the function $G(x)$ graphed above. (The dashed line is $y = x$.)

(a) How many fixed points does the function $G(x)$ have? Mark them on the graph and estimate their values.

3 fixed points, $X_1 \approx -0.5$, $X_2 \approx 0.8$ $X_3 \approx 1.43$

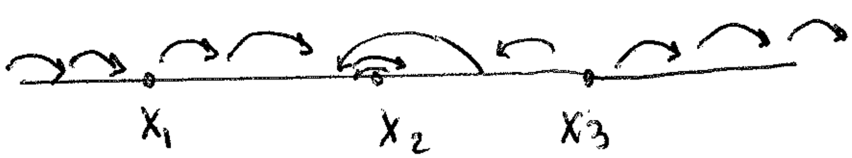
(b) Use graphical analysis to sketch the orbits of -1, 0 and 1 on the graph. Describe the fate of each orbit.

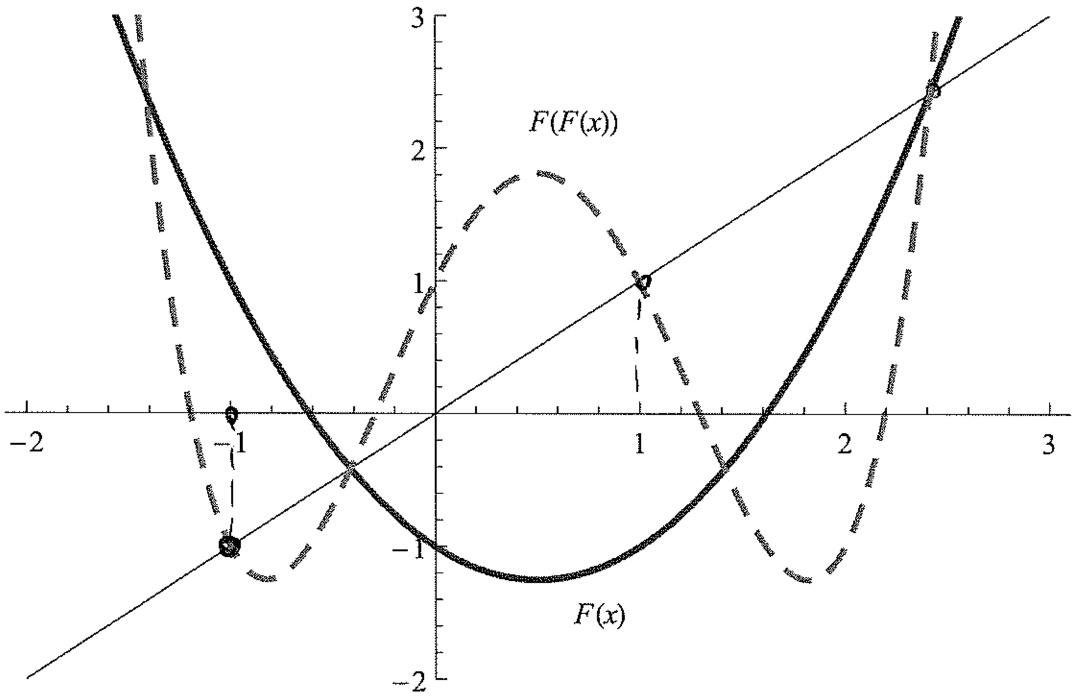
orbit of -1 converges to X_1
orbit of 0 converges to X_2
orbit of 1 converges to X_2

(c) Describe each fixed point as attracting, repelling, or neutral.

X_1 is neutral (one-side attracting)
X_2 is attracting
X_3 is repelling

(d) Sketch the phase portrait of $G(x)$.





4. (10 pts) The functions $F(x)$ (solid curve) and and its second iterate $(F(F(x))$ (dashed curve) are graphed above. (The line is $y = x$.) Does $F(x)$ have any 2-cycles? Estimate their values. Explain your reasoning.

$F(F(x))=x$ has 4 solutions, but two of them are fixed points (also solutions of $F(x)=x$). Remaining two solutions are approximately $\{-1, 1\}$, form 1 2-cycle.

5. **(30 pts)** Consider the one-parameter family $F_c(x) = x^2 + x - c$, where c is real-valued parameter.

(a) How many fixed points does F_c have, depending on c? Find the values of the fixed points.

$x^2 + x - c = x \qquad x^2 = c \qquad x = \pm\sqrt{c}$

$c < 0$ no fixed points

$c = 0$ one fixed point 0

$c > 0$ two fixed points $\pm\sqrt{c}$

(b) Are these fined points attracting, repelling, or neutral? Consider all the cases.

$f'(x) = 2x + 1 \qquad f'(0) = 1$ 0 is neutral f.p.

$f'(\sqrt{c}) = 2\sqrt{c} + 1 > 1$, $\sqrt{c}$ is always repelling.

$f'(-\sqrt{c}) = 1 - 2\sqrt{c} < 1$ Check when $1 - 2\sqrt{c} > -1$

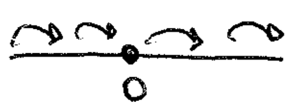
$2 > 2\sqrt{c} \qquad \sqrt{c} < 1 \qquad c < 1$

$-\sqrt{c}$ is attracting for $c \in (0,1)$, neutral for $c = 1$, repelling for $c > 1$

(c) For what value(s) of c does the bifurcation occur? Describe what happens at the bifurcation value and identify the type of the bifurcation. (Note: If you find more than one bifurcation, just describe the easier one.) Sketch the phase portraits before, at, and after the bifurcation value.

$c = 0$ and $c = 1$ Consider $c = 0$: no fixed points for $c < 0$, one neutral for $c = 0$, one attracting and one repelling for $c > 0$. Tangent bifurcation.

$c < 0$ 

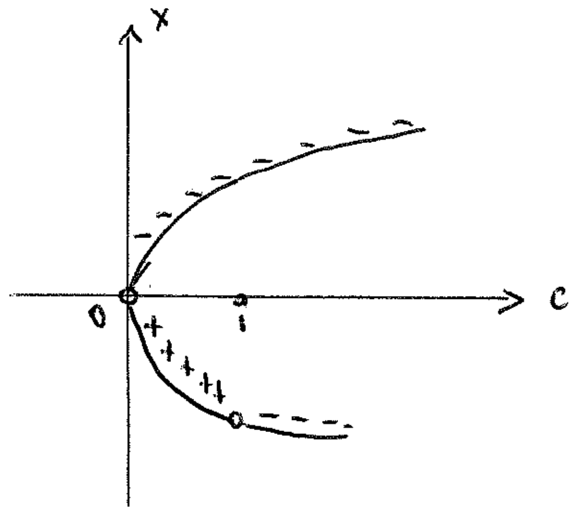
$c = 0$ 

$c > 0$ 

(There is also a period-doubling bifurcation at $c = 1$, but to confirm it you need to find and classify 2-cycles, which is more complicated)

problem 5, continued.

(d) Plot the bifurcation diagram, which will show the location of the fixed points with respect to the parameter value c. (Do not attempt to find and do not plot any cycles.) Use different colors or different curve styles (e.g. solid, dashed, etc.) for attracting, repelling and neutral points. Make a reference which is which.



- - - repelling
o neutral
+ + + + + attracting